

COEFFICIENT BOUNDS FOR NEW SUBCLASSES OF BI-UNIVALENT FUNCTIONS

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ABSTRACT. In the present investigation, we consider two new subclasses $\mathcal{N}_\Sigma^\mu(\alpha, \lambda)$ and $\mathcal{N}_\Sigma^\mu(\beta, \lambda)$ of bi-univalent functions defined in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Besides, we find upper bounds for the second and third coefficients for functions in these new subclasses.

1. INTRODUCTION AND DEFINITIONS

Let \mathcal{A} denote the family of functions f of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Further, let \mathcal{S} denote the class of functions which are univalent in \mathcal{U} .

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1} , defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathcal{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right)$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in \mathcal{U} if both $f(z)$ and $f^{-1}(z)$ are univalent in \mathcal{U} .

Let Σ denote the class of bi-univalent functions in \mathcal{U} given by (1.1).

Lewin [5] investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$ for the functions belonging to Σ . Subsequently, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$. On the other hand, Netanyahu [6] showed that $\max_{f \in \Sigma} |a_2| = \frac{4}{3}$. The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N} \setminus \{1, 2\}$; $\mathbb{N} := \{1, 2, \dots\}$) is still an open problem.

Brannan and Taha [3] (see also [9]) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order α ($0 < \alpha \leq 1$), respectively (see [1]). Thus, following Brannan and Taha [3] (see also [9]), a function $f \in \mathcal{A}$ is in the class $\mathcal{S}_\Sigma^*[\alpha]$ of strongly bi-starlike functions of order α ($0 < \alpha \leq 1$) if each of the following conditions is satisfied:

$$f \in \Sigma \text{ and } \left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, z \in \mathcal{U})$$

and

$$\left| \arg \left(\frac{zg'(w)}{g(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, w \in \mathcal{U}),$$

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where g is the extension of f^{-1} to \mathcal{U} . The classes $\mathcal{S}_{\Sigma}^*(\alpha)$ and $\mathcal{K}_{\Sigma}(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α , corresponding, respectively, to the function classes $\mathcal{S}_{\Sigma}^*(\alpha)$ and $\mathcal{K}_{\Sigma}(\alpha)$, were also introduced analogously. For each of the function classes $\mathcal{S}_{\Sigma}^*(\alpha)$ and $\mathcal{K}_{\Sigma}(\alpha)$, they found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see ([3], [9])).

Recently, Srivastava et al. [8] and Frasin and Aouf [4] have investigated estimate on the coefficients $|a_2|$ and $|a_3|$ for functions in the subclasses $\mathcal{N}_{\Sigma}^1(\alpha, 1)$, $\mathcal{N}_{\Sigma}^1(\alpha, \lambda)$ and $\mathcal{N}_{\Sigma}^1(\beta, 1)$, $\mathcal{N}_{\Sigma}^1(\beta, \lambda)$ which are given by Section 2 and 3, respectively. The main object of the present investigation is to introduce two new subclasses $\mathcal{N}_{\Sigma}^{\mu}(\alpha, \lambda)$ and $\mathcal{N}_{\Sigma}^{\mu}(\beta, \lambda)$ of the function class Σ and to find estimate on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ employing the techniques used earlier by Srivastava et al. [8]. We also extend and improve the aforementioned results of Srivastava et al. [8] and Frasin and Aouf [4]. Various known or new special cases of our results are also pointed out.

Firstly, in order to derive our main results, we need to following lemma.

Lemma 1.1. [7] *If $p \in \mathcal{P}$, then $|c_k| \leq 2$ for each k , where \mathcal{P} is the family of all functions p analytic in \mathcal{U} for which $\Re p(z) > 0$, $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ for $z \in \mathcal{U}$.*

2. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $\mathcal{N}_{\Sigma}^{\mu}(\alpha, \lambda)$

Definition 2.1. *A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{N}_{\Sigma}^{\mu}(\alpha, \lambda)$ if the following conditions are satisfied:*

$$(2.1) \quad f \in \Sigma \text{ and } \left| \arg \left((1 - \lambda) \left(\frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \right) \right| < \frac{\alpha\pi}{2}$$

$$(0 < \alpha \leq 1, \mu \geq 0, z \in \mathcal{U})$$

and

$$(2.2) \quad \left| \arg \left((1 - \lambda) \left(\frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} \right) \right| < \frac{\alpha\pi}{2}$$

$$(0 < \alpha \leq 1, \mu \geq 0, w \in \mathcal{U})$$

where the function g is given by

$$(2.3) \quad g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots$$

Note that for $\lambda = \mu = 1$, the class $\mathcal{N}_{\Sigma}^1(\alpha, 1)$ introduced and studied by Srivastava et al. [8] and for $\mu = 1$, the class $\mathcal{N}_{\Sigma}^1(\alpha, \lambda)$ introduced and studied by Frasin and Aouf [4].

Theorem 2.1. *Let $f(z)$ given by (1.1) be in the class $\mathcal{N}_{\Sigma}^{\mu}(\alpha, \lambda)$, $0 < \alpha \leq 1$, $\lambda \geq 1$ and $\mu \geq 0$. Then*

$$(2.4) \quad |a_2| \leq \frac{2\alpha}{\sqrt{(\lambda + \mu)^2 + \alpha(\mu + 2\lambda - \lambda^2)}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(\lambda + \mu)^2} + \frac{2\alpha}{2\lambda + \mu}.$$

Proof. It follows from (2.1) and (2.2) that

$$(2.5) \quad (1 - \lambda) \left(\frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} = [p(z)]^{\alpha}$$

and

$$(2.6) \quad (1 - \lambda) \left(\frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} = [q(w)]^\alpha$$

where $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ and $q(w) = 1 + q_1 w + q_2 w^2 + \dots$ in \mathcal{P} .

Now, equating the coefficients in (2.5) and (2.6), we have

$$(2.7) \quad (\lambda + \mu) a_2 = \alpha p_1,$$

$$(2.8) \quad (2\lambda + \mu) a_3 + (\mu - 1) \left(\lambda + \frac{\mu}{2} \right) a_2^2 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2,$$

$$(2.9) \quad -(\lambda + \mu) a_2 = \alpha q_1$$

and

$$(2.10) \quad -(2\lambda + \mu) a_3 + (3 + \mu) \left(\lambda + \frac{\mu}{2} \right) a_2^2 = \alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2.$$

From (2.7) and (2.9), we obtain

$$(2.11) \quad p_1 = -q_1$$

and

$$(2.12) \quad 2(\lambda + \mu)^2 a_2^2 = \alpha^2 (p_1^2 + q_1^2).$$

Now, from (2.8), (2.10) and (2.12), we get that

$$\begin{aligned} (\mu + 1) (2\lambda + \mu) a_2^2 &= \alpha (p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2} (p_1^2 + q_1^2) \\ &= \alpha (p_2 + q_2) + \frac{(\alpha - 1)}{\alpha} (\lambda + \mu)^2 a_2^2. \end{aligned}$$

Therefore, we have

$$(2.13) \quad a_2^2 = \frac{\alpha^2 (p_2 + q_2)}{(\lambda + \mu)^2 + \alpha (\mu + 2\lambda - \lambda^2)}.$$

Applying Lemma 1.1 for (2.13), we obtain

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda + \mu)^2 + \alpha (\mu + 2\lambda - \lambda^2)}}$$

which gives us desired estimate on $|a_2|$ as asserted in (2.4).

Next, in order to find the bound on $|a_3|$, by subtracting (2.10) from (2.9), we get

$$(2.14) \quad 2(2\lambda + \mu) a_3 - 2(2\lambda + \mu) a_2^2 = \alpha p_2 + \frac{\alpha(\alpha - 1)}{2} p_1^2 - \left(\alpha q_2 + \frac{\alpha(\alpha - 1)}{2} q_1^2 \right).$$

It follows from (2.11), (2.12) and (2.14) that

$$(2.15) \quad a_3 = \frac{\alpha^2 (p_1^2 + q_1^2)}{2(\lambda + \mu)^2} + \frac{\alpha (p_2 - q_2)}{2(2\lambda + \mu)}$$

Applying Lemma 1.1 for (2.15), we readily get

$$|a_3| \leq \frac{4\alpha^2}{(\lambda + \mu)^2} + \frac{2\alpha}{2\lambda + \mu}.$$

This completes the proof of Theorem 2.1. □

If we take $\mu = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.2. [4] *Let $f(z)$ given by (1.1) be in the class $\mathcal{N}_\Sigma^1(\alpha, \lambda)$, $0 < \alpha \leq 1$ and $\lambda \geq 1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\lambda+1)^2 + \alpha(1+2\lambda-\lambda^2)}} \text{ and } |a_3| \leq \frac{4\alpha^2}{(\lambda+1)^2} + \frac{2\alpha}{2\lambda+1}.$$

If we choose $\lambda = \mu = 1$ in Theorem 2.1, we get the following corollary.

Corollary 2.3. [8] *Let $f(z)$ given by (1.1) be in the class $\mathcal{N}_\Sigma^1(\alpha, 1)$, $0 < \alpha \leq 1$. Then*

$$|a_2| \leq \alpha \sqrt{\frac{2}{\alpha+2}} \text{ and } |a_3| \leq \frac{\alpha(3\alpha+2)}{3}.$$

If we choose $\lambda = \mu + 1 = 1$ in Theorem 2.1, we obtain well-known the class $\mathcal{N}_\Sigma^0(\alpha, 1) = \mathcal{S}_\Sigma^*[\alpha]$ of strongly bi-starlike functions of order α and get the following corollary.

Corollary 2.4. *Let $f(z)$ given by (1.1) be in the class $\mathcal{S}_\Sigma^*[\alpha]$, $0 < \alpha \leq 1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{1+\alpha}} \text{ and } |a_3| \leq \alpha(4\alpha+1).$$

3. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $\mathcal{N}_\Sigma^\mu(\beta, \lambda)$

Definition 3.1. *A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{N}_\Sigma^\mu(\beta, \lambda)$ if the following conditions are satisfied:*

$$(3.1) \quad f \in \Sigma \text{ and } \Re \left((1-\lambda) \left(\frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \right) > \beta$$

$$(0 \leq \beta < 1, \mu \geq 0, \lambda \geq 1, z \in \mathcal{U})$$

and

$$(3.2) \quad \Re \left((1-\lambda) \left(\frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} \right) > \beta$$

$$(0 \leq \beta < 1, \mu \geq 0, \lambda \geq 1, w \in \mathcal{U})$$

where the function g is defined by (2.3).

The class which is satisfy the conditon (3.1) except $f \in \Sigma$ also was studied with other aspects by Zhu [10].

Note that for $\lambda = \mu = 1$, the class $\mathcal{N}_\Sigma^1(\beta, 1)$ introduced and studied by Srivastava et al. [8] and for $\mu = 1$, the class $\mathcal{N}_\Sigma^1(\beta, \lambda)$ introduced and worked by Frasin and Aouf [4].

Theorem 3.1. *Let $f(z)$ given by (1.1) be in the class $\mathcal{N}_\Sigma^\mu(\beta, \lambda)$, $0 \leq \beta < 1$, $\lambda \geq 1$ and $\mu \geq 0$. Then*

$$(3.3) \quad |a_2| \leq \min \left\{ \sqrt{\frac{4(1-\beta)}{(\mu+1)(2\lambda+\mu)}}, \frac{2(1-\beta)}{\lambda+\mu} \right\}$$

and

$$(3.4) \quad |a_3| \leq \begin{cases} \min \left\{ \frac{4(1-\beta)}{(\mu+1)(2\lambda+\mu)}, \frac{4(1-\beta)^2}{(\lambda+\mu)^2} + \frac{2(1-\beta)}{2\lambda+\mu} \right\}; & 0 \leq \mu < 1 \\ \frac{2(1-\beta)}{2\lambda+\mu}; & \mu \geq 1 \end{cases}.$$

Proof. It follows from (3.1) and (3.2) that there exist $p, q \in \mathcal{P}$ such that

$$(3.5) \quad (1 - \lambda) \left(\frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} = \beta + (1 - \beta) p(z)$$

and

$$(3.6) \quad (1 - \lambda) \left(\frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu-1} = \beta + (1 - \beta) q(w)$$

where $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ and $q(w) = 1 + q_1 w + q_2 w^2 + \dots$. As in the proof of Theorem 2.1, by suitably comparing coefficients in (3.5) and (3.6), we get

$$(3.7) \quad (\lambda + \mu) a_2 = (1 - \beta) p_1,$$

$$(3.8) \quad (2\lambda + \mu) a_3 + (\mu - 1) \left(\lambda + \frac{\mu}{2} \right) a_2^2 = (1 - \beta) p_2,$$

$$(3.9) \quad -(\lambda + \mu) a_2 = (1 - \beta) q_1$$

and

$$(3.10) \quad -(2\lambda + \mu) a_3 + (3 + \mu) \left(\lambda + \frac{\mu}{2} \right) a_2^2 = (1 - \beta) q_2.$$

Now, considering (3.7) and (3.9), we obtain

$$(3.11) \quad p_1 = -q_1$$

and

$$(3.12) \quad 2(\lambda + \mu)^2 a_2^2 = (1 - \beta)^2 (p_1^2 + q_1^2).$$

Also, from (3.8) and (3.10), we have

$$(3.13) \quad (\mu + 1)(2\lambda + \mu) a_2^2 = (1 - \beta)(p_2 + q_2).$$

Therefore, from the equalities (3.12) and (3.13) we find that

$$|a_2|^2 \leq \frac{(1 - \beta)^2}{2(\lambda + \mu)^2} (|p_1|^2 + |q_1|^2)$$

and

$$|a_2|^2 \leq \frac{(1 - \beta)}{(\mu + 1)(2\lambda + \mu)} (|p_2| + |q_2|)$$

respectively, and applying Lemma 1.1, we obtain

$$(3.14) \quad |a_2| \leq \frac{2(1 - \beta)}{\lambda + \mu}$$

and

$$(3.15) \quad |a_2| \leq 2 \sqrt{\frac{1 - \beta}{(\mu + 1)(2\lambda + \mu)}}$$

respectively. If we compare the right sides of the inequalities (3.14) and (3.15) we obtain desired estimate on $|a_2|$ as asserted in (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting (3.10) from (3.8), we get

$$(3.16) \quad 2(2\lambda + \mu) a_3 - 2(2\lambda + \mu) a_2^2 = (1 - \beta)(p_2 - q_2),$$

which, upon substitution of the value of a_2^2 from (3.12), yields

$$(3.17) \quad a_3 = \frac{(1 - \beta)^2 (p_1^2 + q_1^2)}{2(\lambda + \mu)^2} + \frac{(1 - \beta)(p_2 - q_2)}{2(2\lambda + \mu)}.$$

Applying Lemma 1.1 for (3.17), we readily get

$$(3.18) \quad |a_3| \leq \frac{4(1-\beta)^2}{(\lambda+\mu)^2} + \frac{2(1-\beta)}{2\lambda+\mu}.$$

On the other hand, by using the equation (3.13) in (3.16), we obtain

$$(3.19) \quad a_3 = \frac{1-\beta}{2(2\lambda+\mu)} \left[\frac{\mu+3}{\mu+1} p_2 + \frac{1-\mu}{\mu+1} q_2 \right].$$

and applying Lemma 1.1 for (3.19), we get

$$(3.20) \quad |a_3| \leq \frac{1-\beta}{2\lambda+\mu} \left[\frac{\mu+3}{\mu+1} + \frac{|1-\mu|}{\mu+1} \right].$$

Now, let us investigate the bounds on $|a_3|$ according to μ .

Case1. We suppose that let $0 \leq \mu < 1$, thus from (3.20)

$$(3.21) \quad |a_3| \leq \frac{4(1-\beta)}{(2\lambda+\mu)(\mu+1)}$$

which is the first part of assertion (3.4).

Case2. We suppose that let $\mu \geq 1$, thus from (3.20) we easily see that

$$(3.22) \quad |a_3| \leq \frac{2(1-\beta)}{2\lambda+1}$$

which is the second part of assertion (3.4). When are compared the right sides of inequalities (3.18) and (3.22) we see that the right side of (3.22) smaller than the right side of (3.18).

This completes the proof of Theorem 3.1. \square

If we write $\mu = 1$ in first parts of assertions (3.3) and (3.4) of Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let $f(z)$ given by (1.1) be in the class $\mathcal{N}_\Sigma^1(\beta, \lambda)$, $0 \leq \beta < 1$ and $\lambda \geq 1$. Then*

$$|a_2| \leq \min \left\{ \sqrt{\frac{2(1-\beta)}{2\lambda+1}}, \frac{2(1-\beta)}{\lambda+1} \right\} \quad \text{and} \quad |a_3| \leq \frac{2(1-\beta)}{2\lambda+1}.$$

If we choose $\lambda = \mu = 1$ in first parts of assertions (3.3) and (3.4) of Theorem 3.1, we have the following corollary.

Corollary 3.3. *Let $f(z)$ given by (1.1) be in the class $\mathcal{N}_\Sigma^1(\beta, 1)$, $0 \leq \beta < 1$. Then*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{3}}; & 0 \leq \beta < \frac{1}{3} \\ 1-\beta & \frac{1}{3} \leq \beta < 1 \end{cases} \quad \text{and} \quad |a_3| \leq \frac{2(1-\beta)}{3}.$$

If we take $\lambda = \mu + 1 = 1$ in Theorem 3.1, we obtain well-known the class $\mathcal{N}_\Sigma^0(\beta, 1) = \mathcal{S}_\Sigma^*(\beta)$ of bi-starlike functions of order β and get the following corollary.

Corollary 3.4. *Let $f(z)$ given by (1.1) be in the class $\mathcal{S}_\Sigma^*(\beta)$, $0 \leq \beta < 1$. Then*

$$|a_2| \leq \sqrt{2(1-\beta)} \quad \text{and} \quad |a_3| \leq \begin{cases} 2(1-\beta); & 0 \leq \beta < \frac{3}{4} \\ (1-\beta)(5-4\beta); & \frac{3}{4} \leq \beta < 1 \end{cases}.$$

REFERENCES

- [1] D.A. Brannan, J. Clunie, W.E. Kirwan, Coefficient estimates for a class of starlike functions, *Canad. J. Math.* 22 (1970) 476–485.
- [2] D.A. Brannan, J.G. Clunie (Eds.), *Aspects of Contemporary Complex Analysis* (Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham; July 1 20, 1979), Academic Press, New York and London, 1980.
- [3] D.A. Brannan, T.S. Taha, On some classes of bi-univalent functions, in: S.M. Mazhar, A. Hamoui, N.S. Faour (Eds.), *Math. Anal. and Appl.*, Kuwait; February 18–21, 1985, in: *KFAS Proceedings Series*, vol. 3, Pergamon Press, Elsevier Science Limited, Oxford, 1988, pp. 53–60. see also *Studia Univ. Babeş-Bolyai Math.* 31 (2) (1986) 70–77.
- [4] B.A. Frasin, M.K. Aouf, New subclasses of bi-univalent functions, *Appl. Math. Lett.* 24 (2011) 1569–1573.
- [5] M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc.* 18 (1967) 63–68.
- [6] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, *Arch. Rational Mech. Anal.* 32 (1969) 100–112.
- [7] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.
- [8] H.M. Srivastava, A.K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.* 23 (2010) 1188–1192.
- [9] T.S. Taha, *Topics in univalent function theory*, Ph.D. Thesis, University of London, 1981.
- [10] Y. Zhu, Some starlikeness criteria for analytic functions, *J. Math. Anal. Appl.* 335 (2007) 1452–1459.

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